Linear Diophantine Equations (LDEs)

Definition 1
An equation of the form

\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \]  

(1)

with \( a_1, a_2, \ldots, a_n, b \) integers, is called a linear Diophantine equation (LDE).

Theorem 2
The LDE

\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \]

has a solution \( x_1, \ldots, x_n \in \mathbb{Z} \) if and only if \( \gcd(a_1, a_2, \ldots, a_n) | b \).
Quadratic Diophantine Equations (QDEs)

Definition 3
An equation of the form

\[ \sum_{i,j=1}^{n} a_{ij}x_i x_j = b \]  

(2)

with \( a_{ij}, b \) integers, is called a quadratic Diophantine equation (QDE).

Example 4 (Pythagorean Equations)
The equation

\[ x^2 + y^2 = z^2 \]

is a QDE. Any solution \((x, y, z)\) of this equation for integers \(x, y, z\) is called a Pythagorean triple.
Consider the Pythagorean equation:

\[ x^2 + y^2 = z^2. \]  \hspace{1cm} (3)

- A solution \((x_0, y_0, z_0)\) of Eq. (3) where \(x_0, y_0, z_0\) are pairwise relatively prime is called a primitive solution.
- If \((x_0, y_0, z_0)\) is a solution of Eq. (3) then so are

\[ (\pm x_0, \pm y_0, \pm z_0) \text{ and } (kx_0, ky_0, kz_0). \]

- Therefore we are most interested in solutions \((x, y, z)\) of Eq. (3) with all components positive.
Pythagorean Equations

Theorem 5
Any primitive solution of

\[ x^2 + y^2 = z^2 \]

is of the form

\[ x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2 \] \hspace{1cm} (4)

Where \( m, n \geq 1 \) are relatively prime positive integers.
Pell’s Equation

Definition 6
Pell’s equation has the form

\[ x^2 - dy^2 = 1 \]  \hspace{1cm} (5)

where \( d \) not a perfect square.

Definition 7
We say that \((x_0, y_0)\) is a fundamental solution of Pell’s equation if \(x_0, y_0\) are positive integers that are minimal amongst all solutions.
The equation has the fundamental solution $(x_0, y_0) = (3, 2)$.
Pell’s Equation

Theorem 8

Pell’s equation has infinitely many solutions. Given the solution \((x_0, y_0)\) the solution \((x_{n+1}, y_{n+1})\) is given by

\[
\begin{align*}
x_{n+1} &= x_0 x_n + d y_0 y_n, \quad x_1 = x_0, \quad n \geq 1 \\
y_{n+1} &= y_0 x_n + x_0 y_n, \quad y_1 = y_0, \quad n \geq 1
\end{align*}
\]

Example 9

The equation \(x^2 - 2y^2 = 1\), has the fund. sol. \((x_0, y_0) = (3,2)\). So

\[
x_2 = x_0^2 + d y_0^2 = 9 + 2.4 = 17, \quad y_2 = y_0 x_0 + x_0 y_0 = 6 + 6 = 12
\]

is also a solution: \(17^2 - 2 \cdot 12^2 = 1\).
General Solution of Pell’s Equation

Theorem 10

Let Pell’s equation \( x^2 - dy^2 = 1 \), have the fundamental solution \((x_0, y_0)\). Then \((x_n, y_n)\) is also a solution, given by

\[
\begin{align*}
x_n &= \frac{1}{2}[(x_0 + y_0\sqrt{d})^n + (x_0 - y_0\sqrt{d})^n] \\
y_n &= \frac{1}{2\sqrt{d}}[(x_0 + y_0\sqrt{d})^n - (x_0 - y_0\sqrt{d})^n]
\end{align*}
\]

(7)

Example 11

Solve \( x^2 - 2y^2 = 1 \). The fund. sol. is \((3,2)\). The general solution is:

\[
x_n = \frac{1}{2}[(3+2\sqrt{2})^n+(3-2\sqrt{2})^n], \quad y_n = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^n-(3-2\sqrt{2})^n]
\]
Definition 12
The general Pell’s equation has the form

\[ ax^2 - by^2 = 1 \] (8)

where \( ab \) not a perfect square.

The equation

\[ u^2 - abv^2 = 1 \] (9)

is called the Pell’s resolvent of Eq. (8)
The General Form of Pell’s Equation

Theorem 13

Let

\[ ax^2 - by^2 = 1 \]

have an integral solution. Let \((A, B)\) solution for least positive \(A, B\). The general solution is

\[ x_n = Au_n + bBv_n \]

\[ y_n = Bu_n + aAv_n \]

(10)

Where \((u_n, v_n)\) is the general solution of Pell’s resolvent

\[ u^2 - abv^2 = 1. \]
The General Form of Pell’s Equation

**Example 14**

Solve

\[ 6x^2 - 5y^2 = 1 \] \hspace{1cm} (11)

The fund. sol. is \((x, y) = (A, B) = (1, 1)\). The resolvent is \(u^2 - 30v^2 = 1\), with fund. sol. \((u_0, v_0) = (11, 2)\). The general solution of the resolvent is

\[
\begin{align*}
 u_n &= \frac{1}{2} \left[ (11 + 2\sqrt{30})^n + (11 - 2\sqrt{30})^n \right] \\
 v_n &= \frac{1}{2\sqrt{30}} \left[ (11 + 2\sqrt{30})^n - (11 - 2\sqrt{30})^n \right]
\end{align*}
\]

The general solution of Eq. (11) is

\[
x_n = u_n + 5v_n, \quad y_n = u_n + 6v_n
\]
Problem 1

Find all integers \( n \geq 1 \) such that \( 2n + 1 \) and \( 3n + 1 \) are both perfect squares.
Problem 1

*Find all integers* \( n \geq 1 \) *such that* \( 2n + 1 \) *and* \( 3n + 1 \) *are both perfect squares.*

Observe that

\[
2n + 1 = x^2, \quad 3n + 1 = y^2 \implies 3x^2 - 2y^2 = 1,
\]

with \( 3 \cdot 2 = 6 \) not a square in \( \mathbb{Z} \).

So solving this amounts to solving the general form of Pell’s equation.
The Negative Pell’s Equation

Definition 15
The negative Pell’s equation has the form

\[ x^2 - dy^2 = -1 \]  \hspace{1cm} (12)

where \( d \) not a perfect square.
The Negative Pell’s Equation

Definition 15
The negative Pell’s equation has the form

\[ x^2 - dy^2 = -1 \]  \hspace{1cm} (12)

where \( d \) not a perfect square.

Theorem 16
Let \((A, B)\) be the smallest positive solution to Eq. (12). Then the general solution to Eq. (12) is given by

\[
\begin{align*}
    x_n &= Au_n + dBv_n \\
    y_n &= Au_n + Bv_n
\end{align*}
\] \hspace{1cm} (13)

where \((u_n, v_n)\) is the general solution of \( u^2 - dv^2 = 1 \).
Problem 2

Find all pairs \((k, m)\) such that

\[ 1 + 2 + \cdots + k = (k + 1) + (k + 2) + \cdots + m. \]
Problem 2

Find all pairs \((k, m)\) such that

\[
1 + 2 + \cdots + k = (k + 1) + (k + 2) + \cdots + m.
\]

Adding \(1 + 2 + \cdots + k\) to both sides of the above equality we get

\[
2k(k + 1) = m(m + 1) \iff (2m + 1)^2 - 2(2k + 1)^2 = -1.
\]
Problem 2

Find all pairs \((k, m)\) such that

\[1 + 2 + \cdots + k = (k + 1) + (k + 2) + \cdots + m.\]

Adding \(1 + 2 + \cdots + k\) to both sides of the above equality we get

\[2k(k + 1) = m(m + 1) \iff (2m + 1)^2 - 2(2k + 1)^2 = -1.\]

The associated negative Pell’s equation is \(x^2 - 2y^2 = -1\) with the minimal solution \((A, B) = (1, 1)\).
Problem 3 (Romanian M. Olympiad, 1999)

Show that the equation $x^2 + y^3 + z^3 = t^4$ has infinitely many solutions $x, y, z, t, \in \mathbb{Z}$ with the greatest common divisor 1.
Show that the equation \( x^2 + y^3 + z^3 = t^4 \) has infinitely many solutions \( x, y, z, t \in \mathbb{Z} \) with the greatest common divisor 1.

Start from the equality

\[
[1^3 + 2^3 + \cdots + (n-2)^3] + (n-1)^3 + n^3 = \left( \frac{n(n+1)}{2} \right)^2
\]

\[
\left[ \frac{(n-2)(n-1)}{2} \right]^2 + (n-1)^3 + n^3 = \left( \frac{n(n+1)}{2} \right)^2.
\]
Problem 3 (Romanian M. Olympiad, 1999)

Show that the equation $x^2 + y^3 + z^3 = t^4$ has infinitely many solutions $x, y, z, t, \in \mathbb{Z}$ with the greatest common divisor 1.

Start from the equality

$$[1^3 + 2^3 + \cdots + (n - 2)^3] + (n - 1)^3 + n^3 = \left( \frac{n(n + 1)}{2} \right)^2$$

$$\left[ \frac{(n - 2)(n - 1)}{2} \right]^2 + (n - 1)^3 + n^3 = \left( \frac{n(n + 1)}{2} \right)^2.$$

Do there exist infinitely many integers $n \geq 1$ such that $\frac{n(n+1)}{2}$ is a perfect square?
Prove that the equation \( x^2 + y^3 + z^3 = t^4 \) has infinitely many solutions \( x, y, z, t \in \mathbb{Z} \) with the greatest common divisor 1.

Start from the equality

\[
[1^3 + 2^3 + \cdots + (n - 2)^3] + (n - 1)^3 + n^3 = \left( \frac{n(n + 1)}{2} \right)^2
\]

Do there exist infinitely many integers \( n \geq 1 \) such that \( \frac{n(n+1)}{2} \) is a perfect square?

\[
n(n + 1) = n^2 + n = 2m^2 \iff 4n^2 + 4n = 8m^2
\]

\[
\iff (2n + 1)^2 - 2(2m)^2 = 1
\]

This is Pell’s equation, which has infinitely many solutions.
Problem 4 (Irish M. Olympiad, 1995)

Determine all integers $a$ such that the equation $x^2 + axy + y^2 = 1$ has infinitely many solutions.
Problem 4 (Irish M. Olympiad, 1995)

Determine all integers $a$ such that the equation $x^2 + axy + y^2 = 1$ has infinitely many solutions.

Rewrite the given equation in the form

$$(2x + ay)^2 - (a^2 - 4)y^2 = 4$$  (14)
Problem 4 (Irish M. Olympiad, 1995)

Determine all integers a such that the equation \( x^2 + axy + y^2 = 1 \) has infinitely many solutions.

Rewrite the given equation in the form

\[
(2x + ay)^2 - (a^2 - 4)y^2 = 4 \quad (14)
\]

1. If \( a^2 - 4 < 0 \) then we have a finite number of solutions.

2. If \( a^2 - 4 = 0 \) the equation becomes \( 2x + ay = \pm 2 \) with infinitely many solutions.

3. If \( a^2 - 4 > 0 \), then \( a^2 - 4 \) cannot be a perfect square and so the Pell's equation

\[
u^2 - (a^2 - 4)v^2 = 1
\]

has infinitely many solutions. Letting \( x = u - av \), \( y = 2v \), we also have infinitely many solutions for \( a^2 - 4 \geq 0 \).
Problem 4 (Irish M. Olympiad, 1995)

*Determine all integers* $a$ *such that the equation* $x^2 + axy + y^2 = 1$
*has infinitely many solutions.*

Rewrite the given equation in the form

$$(2x + ay)^2 - (a^2 - 4)y^2 = 4 \quad (14)$$

1. If $a^2 - 4 < 0$ then we have a finite number of solutions.
2. If $a^2 - 4 = 0$ the equation becomes $2x + ay = \pm 2$ with infinitely many solutions.
Problem 4 (Irish M. Olympiad, 1995)

*Determine all integers $a$ such that the equation $x^2 + axy + y^2 = 1$ has infinitely many solutions.*

Rewrite the given equation in the form

$$(2x + ay)^2 - (a^2 - 4)y^2 = 4 \quad (14)$$

1. If $a^2 - 4 < 0$ then we have a finite number of solutions.

2. If $a^2 - 4 = 0$ the equation becomes $2x + ay = \pm 2$ with infinitely many solutions.

3. If $a^2 - 4 > 0$, then $a^2 - 4$ cannot be a perfect square and so the Pell’s equation $u^2 - (a^2 - 4)v^2 = 1$ has infinitely many solutions. Letting $x = u - av, y = 2v$, we also have infinitely many solutions for $a^2 - 4 \geq 0$
Training Problem 5

Problem 5 (Bulgarian M. Olympiad, 1999)

Solve $x^3 = y^3 + 2y^2 + 1$ for integers $x, y$. 

Problem 5 (Bulgarian M. Olympiad, 1999)

Solve \( x^3 = y^3 + 2y^2 + 1 \) for integers \( x, y \).

If \( y^2 + 3y > 0 \) then

\[
y^3 < x^3 = y^3 + 2y^2 + 1 < (y^3 + 2y^2 + 1) + (y^2 + 3y) = (y + 1)^3,
\]

which is impossible.
Problem 5 (Bulgarian M. Olympiad, 1999)

Solve \( x^3 = y^3 + 2y^2 + 1 \) for integers \( x, y \).

If \( y^2 + 3y > 0 \) then

\[
y^3 < x^3 = y^3 + 2y^2 + 1 < (y^3 + 2y^2 + 1) + (y^2 + 3y) = (y + 1)^3,
\]

which is impossible.

Therefore

\[
y^2 + 3y \leq 0 \implies y = 0, -1, -2, -3.
\]

The solution set is \((1, 0), (1, -2), (-2, -3)\).
Problem 6

Find positive integers $x, y, z$ such that $xy + yz + zx - xyz = 2$
Problem 6

Find positive integers $x, y, z$ such that $xy + yz + zx - xyz = 2$

We may assume that $x \leq y \leq z$. 
Problem 6

Find positive integers $x, y, z$ such that $xy + yz + zx - xyz = 2$

We may assume that $x \leq y \leq z$.

1. If $x = 1$ then the equation is $y + z = 2 \implies (x, y, z) = (1, 1, 1)$
Problem 6

Find positive integers \(x, y, z\) such that \(xy + yz + zx - xyz = 2\)

We may assume that \(x \leq y \leq z\).

1. If \(x = 1\) then the equation is \(y + z = 2\) \(\implies (x, y, z) = (1, 1, 1)\)

2. If \(x = 2\) then the equation is
   \[2y + 2z - yz = 2 = (z - 2)(y - 2) \implies z = 4, y = 3.\]
Problem 6

Find positive integers $x, y, z$ such that $xy + yz + zx - xyz = 2$

We may assume that $x \leq y \leq z$.

1. If $x = 1$ then the equation is $y + z = 2 \implies (x, y, z) = (1, 1, 1)$

2. If $x = 2$ then the equation is
   $2y + 2z - yz = 2 = (z - 2)(y - 2) \implies z = 4, y = 3$.

3. If $x \geq 3$ then $x, y, z, \geq 3$ which yield
   
   \begin{align*}
   xyz &\geq 3xy \\
   xyz &\geq 3yz \\
   xyz &\geq 3zx
   \end{align*}
Problem 6

Find positive integers $x, y, z$ such that $xy + yz + zx - xyz = 2$

We may assume that $x \leq y \leq z$.

1. If $x = 1$ then the equation is $y + z = 2 \implies (x, y, z) = (1, 1, 1)$

2. If $x = 2$ then the equation is
   
   $2y + 2z - yz = 2 = (z - 2)(y - 2) \implies z = 4, y = 3$.

3. If $x \geq 3$ then $x, y, z, \geq 3$ which yield

   \[
   \begin{align*}
   xyz & \geq 3xy \\
   xyz & \geq 3yz \\
   xyz & \geq 3zx
   \end{align*}
   \]

   Adding the above relations it follows that

   
   
   $xyz \geq xy + yz + zx \implies xy + yz + zx - xyz < 0 \neq 2$. 


Problem 7

Find the positive integers $x, y, z$ such that $3^x + 4^y = z^2$. 
Problem 7

*Find the positive integers* $x, y, z$ *such that* $3^x + 4^y = z^2$.

$$3^x = z^2 - 4^y = (z - 2^y)(z + 2^y).$$

Then

$$z - 2^y = 3^m \text{ and } z + 2^y = 3^n, \quad m > n \geq 0, \quad m + n = x.$$
Problem 7

Find the positive integers $x, y, z$ such that $3^x + 4^y = z^2$. 

$$3^x = z^2 - 4^y = (z - 2^y)(z + 2^y).$$

Then

$$z - 2^y = 3^m \text{ and } z + 2^y = 3^n, \ m > n \geq 0, \ m + n = x.$$ 

Subtracting,

$$2^{y+1} = 3^n - 3^m = 3^m(3^{n-m} - 1) \implies 3^m = 1, \ n = x \implies 3^n - 1 = 2^{y+1}$$
Problem 7

Find the positive integers $x, y, z$ such that $3^x + 4^y = z^2$.

$$3^x = z^2 - 4^y = (z - 2^y)(z + 2^y).$$

Then

$$z - 2^y = 3^m \text{ and } z + 2^y = 3^n, \quad m > n \geq 0, \quad m + n = x.$$ 

Subtracting,

$$2^{y+1} = 3^n - 3^m = 3^m(3^{n-m} - 1)$$

$$\implies 3^m = 1, \quad n = x \implies 3^n - 1 = 2^{y+1}$$

1. If $y = 0$, then $n = x = 1$ and $z = 2$.
2. If $y \geq 1$ then $x = n = 2, y = 2, z = 3^n - 2^y = 5$. 
Problem 8

Find the positive integers $x, y, z$ such that $3^x - 1 = y^z$. 
Problem 8

*Find the positive integers* $x, y, z$ *such that* $3^x - 1 = y^z$.

If $z$ is even we get a contradiction. So $z = 2k + 1$. 
Problem 8

Find the positive integers $x, y, z$ such that $3^x - 1 = y^z$. If $z$ is even we get a contradiction. So $z = 2k + 1$. Now

$$3^x = y^z + 1 = y^{2k+1} + 1 = (y+1)(y^{2k} - y^{2k-1} + y^{2k-2} - \cdots + y^2 - y + 1).$$
Problem 8

Find the positive integers \(x, y, z\) such that \(3^x - 1 = y^z\).

If \(z\) is even we get a contradiction. So \(z = 2k + 1\). Now

\[
3^x = y^z + 1 = y^{2k+1} + 1 = (y+1)(y^{2k} - y^{2k-1} + y^{2k-2} - \cdots + y^2 - y + 1).
\]

Then \(y \equiv -1 \mod 3\).
Problem 8

Find the positive integers $x, y, z$ such that $3^x - 1 = y^z$.

If $z$ is even we get a contradiction. So $z = 2k + 1$. Now

$$3^x = y^z + 1 = y^{2k+1} + 1 = (y+1)(y^{2k} - y^{2k-1} + y^{2k-2} - \cdots + y^2 - y + 1).$$

Then $y \equiv -1 \mod 3$.

$$y^{2k} - y^{2k-1} + \cdots + y^2 - y + 1 \equiv 1 + 1 + \cdots + 1 \equiv (2k+1) \equiv 0 \mod 3.$$ 

Therefore $z = 2k + 1 = 3p$, some $p$:

$$3^x = y^{3p} + 1 = (y^p + 1)(y^{2p} - y^p + 1) \implies y^p + 1 = 3^s.$$ 

$$3^x = 1 + y^{3p} = 1 + (3^s - 1)^3$$

$$= 3^{3s} - 3 \cdot 3^{2s} + 3 \cdot 3^s$$

$$= 3^{s+1}(3^{2s-1} - 3^5 + 1)$$

$$\implies 3^{2s-1} - 3^s = 0 \implies s = 1$$

$$\implies y^p = 3^s - 1 = 2 \implies y = 2, p = 1, x = 2, z = 3.$$
Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers \(a, b, c \geq 1\) such that \(a^b + 1 = (a + 1)^c\).
Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers \(a, b, c \geq 1\) such that \(a^b + 1 = (a + 1)^c\)

1. \(b = c = 1, a \geq 1\) is a solution. Let \(b \geq 2\).
Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers $a, b, c \geq 1$ such that $a^b + 1 = (a + 1)^c$

1. $b = c = 1, a \geq 1$ is a solution. Let $b \geq 2$.
2. $a^b + 1 = (a + 1)^c \equiv (-1)^b + 1 \equiv 0 \mod a + 1 \implies b$ odd
Training Problem 9

Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers $a, b, c \geq 1$ such that $a^b + 1 = (a + 1)^c$

1. $b = c = 1$, $a \geq 1$ is a solution. Let $b \geq 2$.

2. $a^b + 1 = (a + 1)^c \equiv (-1)^b + 1 \equiv 0 \mod a + 1 \implies b$ odd

3. $(a + 1 - 1)^b + 1 \equiv b(a + 1) \equiv 0 \mod (a + 1)^2 \implies a$ even
Training Problem 9

Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers $a, b, c \geq 1$ such that $a^b + 1 = (a + 1)^c$

1. $b = c = 1, a \geq 1$ is a solution. Let $b \geq 2$.
2. $a^b + 1 = (a + 1)^c \equiv (-1)^b + 1 \equiv 0 \mod a + 1 \implies b$ odd
3. $(a + 1 - 1)^b + 1 \equiv b(a + 1) \equiv 0 \mod (a + 1)^2 \implies a$ even
4. $a^b + 1 \equiv 1 \equiv (a + 1)^c \equiv ca + 1 \mod a^2 \implies a|c \implies c$ even
Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers \( a, b, c \geq 1 \) such that \( a^b + 1 = (a + 1)^c \)

1. \( b = c = 1, a \geq 1 \) is a solution. Let \( b \geq 2 \).
2. \( a^b + 1 = (a + 1)^c \equiv (-1)^b + 1 \equiv 0 \mod a + 1 \implies b \) odd
3. \( (a + 1 - 1)^b + 1 \equiv b(a + 1) \equiv 0 \mod (a + 1)^2 \implies a \) even
4. \( a^b + 1 \equiv 1 \equiv (a + 1)^c \equiv ca + 1 \mod a^2 \implies a \mid c \implies c \) even
5. \( (2x)^b = (a + 1)^{2y} - 1 = [(a + 1)^y - 1][(a + 1)^y + 1] \)
Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers $a, b, c \geq 1$ such that $a^b + 1 = (a + 1)^c$

1. $b = c = 1, a \geq 1$ is a solution. Let $b \geq 2$.
2. $a^b + 1 = (a + 1)^c \equiv (-1)^b + 1 \equiv 0 \mod a + 1 \implies b$ odd
3. $(a + 1 - 1)^b + 1 \equiv b(a + 1) \equiv 0 \mod (a + 1)^2 \implies a$ even
4. $a^b + 1 \equiv 1 \equiv (a + 1)^c \equiv ca + 1 \mod a^2 \implies a|c \implies c$ even
5. $(2x)^b = (a + 1)^{2y} - 1 = [(a + 1)^y - 1][(a + 1)^y + 1]$
6. $gcd((a + 1)^y - 1, (a + 1)^y + 1) = 2$
Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers \(a, b, c \geq 1\) such that \(a^b + 1 = (a + 1)^c\)

1. \(b = c = 1, a \geq 1\) is a solution. Let \(b \geq 2\).
2. \(a^b + 1 = (a + 1)^c \equiv (-1)^b + 1 \equiv 0 \mod a + 1 \implies b\) odd
3. \((a + 1 - 1)^b + 1 \equiv b(a + 1) \equiv 0 \mod (a + 1)^2 \implies a\) even
4. \(a^b + 1 \equiv 1 \equiv (a + 1)^c \equiv ca + 1 \mod a^2 \implies a|c \implies c\) even
5. \((2x)^b = (a + 1)^{2y} - 1 = [(a + 1)^y - 1][(a + 1)^y + 1]
6. \(gcd((a + 1)^y - 1, (a + 1)^y + 1) = 2\)
7. \(x|(a + 1)^y - 1 = (2x + 1)^y - 1 \implies (a + 1)^y - 1 = 2x^b\)
Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers $a, b, c \geq 1$ such that $a^b + 1 = (a + 1)^c$

1. $b = c = 1, a \geq 1$ is a solution. Let $b \geq 2$.
2. $a^b + 1 = (a + 1)^c \equiv (-1)^b + 1 \equiv 0 \pmod{a + 1} \implies b$ odd
3. $(a + 1 - 1)^b + 1 \equiv b(a + 1) \equiv 0 \pmod{(a + 1)^2} \implies a$ even
4. $a^b + 1 \equiv 1 \equiv (a + 1)^c \equiv ca + 1 \pmod{a^2} \implies a|c \implies c$ even
5. $(2x)^b = (a + 1)^{2y} - 1 = [(a + 1)^y - 1][(a + 1)^y + 1]$
6. $gcd((a + 1)^y - 1, (a + 1)^y + 1) = 2$
7. $x|(a + 1)^y - 1 = (2x + 1)^y - 1 \implies (a + 1)^y - 1 = 2x^b$
8. $2^{b-1} = (a + 1)^y + 1 > (a + 1)^y - 1 = 2x^b \implies x = 1$
Find all positive integers $a, b, c \geq 1$ such that $a^b + 1 = (a + 1)^c$

1. $b = c = 1, a \geq 1$ is a solution. Let $b \geq 2$.
2. $a^b + 1 = (a + 1)^c \equiv (-1)^b + 1 \equiv 0 \mod a + 1 \implies b$ odd
3. $(a + 1 - 1)^b + 1 \equiv b(a + 1) \equiv 0 \mod (a + 1)^2 \implies a$ even
4. $a^b + 1 \equiv 1 \equiv (a + 1)^c \equiv ca + 1 \mod a^2 \implies a|c \implies c$ even
5. $(2x)^b = (a + 1)^{2y} - 1 = [(a + 1)^y - 1][(a + 1)^y + 1]$
6. $gcd((a + 1)^y - 1, (a + 1)^y + 1) = 2$
7. $x|(a + 1)^y - 1 = (2x + 1)^y - 1 \implies (a + 1)^y - 1 = 2x^b$
8. $2^{b-1} = (a + 1)^y + 1 > (a + 1)^y - 1 = 2x^b \implies x = 1$
9. The only other solution is $a = 2, b = c = 3$. 