# MATH00040 Formula Sheet

## Matrices and Vectors

Matrix inverses: If $ad - bc \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Determinants:

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Length of a vector: If $\mathbf{a} = (a_1, a_2, a_3)$, then $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Dot product: If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Angle between vectors: $\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|}\right)$.

Cross product: $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$, where $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$.

Eigenvalues: To find the eigenvalues of $A$, solve $\det(A - \lambda I) = 0$.

Eigenvectors: To find the eigenvector $\mathbf{v}$ corresponding to the eigenvalue $\lambda$, solve $A\mathbf{v} = \lambda\mathbf{v}$, where $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

## Complex numbers

Real part of a complex number: $\text{Re}(a + bi) = a$.

Imaginary part of a complex number: $\text{Im}(a + bi) = b$.

Modulus of a complex number: $|a + bi| = \sqrt{a^2 + b^2}$.

Complex conjugate of a complex number: $\overline{a + bi} = a - bi$.

Dividing complex numbers: $\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}$.

Polar form: $a+bi = r(\cos(\theta) + i\sin(\theta))$, where $r = |a+bi|$ and we find $\theta$ as follows.
First calculate $\phi = \tan^{-1}\left(\left|\frac{b}{a}\right|\right)$. Then

$$\theta = \phi \text{ if } a > 0 \text{ and } b > 0, \qquad \theta = \pi - \phi \text{ if } a < 0 \text{ and } b > 0,$$
$$\theta = \phi - \pi \text{ if } a < 0 \text{ and } b < 0 \quad \text{and} \quad \theta = -\phi \text{ if } a > 0 \text{ and } b < 0.$$

Common values of $\tan^{-1}$:

$$\tan^{-1}(0) = 0, \quad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}, \quad \tan^{-1}(1) = \frac{\pi}{4}, \quad \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}.$$

De Moivre's formula: $[r(\cos(\theta) + i\sin(\theta))]^n = r^n(\cos(n\theta) + i\sin(n\theta))$.
Roots of complex numbers: The $n$'th roots of $r(\cos(\theta) + i\sin(\theta))$ are

$$z_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right) \qquad k = 0, 1, \ldots, n-1.$$

## Differential Calculus

| $f(x)$ | $f'(x)$ | Comments |
|---|---|---|
| $c$ | $0$ | Here $c$ is any real number |
| $x^n$ | $nx^{n-1}$ | |
| $e^{ax}$ | $ae^{ax}$ | |
| $\ln(ax)$ | $\frac{1}{x}$ | Here we must have $ax > 0$ |
| $\sin(ax)$ | $a\cos(ax)$ | |
| $\cos(ax)$ | $-a\sin(ax)$ | Note the change of sign |

Table 1: Some common derivatives

The product rule for differentiation: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
The quotient rule for differentiation: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.
The chain rule for differentiation: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.
The critical points of a function $f$ occur at points $x$ where $f'(x) = 0$.
The solutions of the equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
Classifying critical points: Local minima occur where $f''(x) > 0$.
Local maxima occur where $f''(x) < 0$.
The Newton-Raphson method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

## Integral Calculus

| $f(x)$ | $\int f(x)\,dx$ | Comments |
|---|---|---|
| $k$ | $kx + c$ | Here $k$ is any real number |
| $x^n$ | $\frac{1}{n+1}x^{n+1} + c$ | Here we must have $n \neq -1$ |
| $\frac{1}{x}$ | $\ln(x) + c$ | Here we must have $x > 0$ |
| $e^{ax}$ | $\frac{1}{a}e^{ax} + c$ | |
| $\sin(ax)$ | $-\frac{1}{a}\cos(ax) + c$ | Note the change of sign |
| $\cos(ax)$ | $\frac{1}{a}\sin(ax) + c$ | |

Table 2: Some common integrals

Evaluating definite integrals: If $F$ is an antiderivative of $f$ then

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Integration by substitution: $dx = \dfrac{du}{du/dx}$.

Integration by parts: $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$.

Volume of solid of revolution: $V = \pi \int_a^b f(x)^2\,dx$.

## Probability

Sum rule for probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Binomial distribution: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

Poisson distribution: $P(X = k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$.

Normal distribution: $P(a \leqslant X \leqslant b) = P\left(\dfrac{a-\mu}{\sigma} \leqslant Z \leqslant \dfrac{b-\mu}{\sigma}\right)$.