Lipschitz free spaces, bases and approximation properties

1. Introduction

The study of **Lipschitz-free** (free) spaces is an exciting and expanding area of mathematical research. The aim of this project is to improve our knowledge of free spaces by addressing a range of pertinent questions concerning their structure.

Non-linear Banach space theory seeks to describe the extent to which the linear structure of a Banach space is preserved by certain classes of non-linear maps, e.g. Lipschitz isomorphisms or Lipschitz embeddings (and quotients). The subject originated with the 1932 Mazur-Ulam Theorem (**surjective** isometries between normed spaces must be affine), but it really came to prominence after the publication of the book of Benyamini and Lindenstrauss in 2000 [1].

The **free space** $\mathcal{F}(M)$ (referred to as Arens–Eells spaces in [12]) is a Banach space obtained naturally from a metric space M, whose key property is that every Lipschitz map between metric spaces M and N extends to a linear map (having the same Lipschitz constant) between $\mathcal{F}(M)$ and $\mathcal{F}(N)$. In a seminal work (81 citations according to MathSciNet), Godefroy and Kalton used free spaces to prove a deep generalization of the Mazur-Ulam theorem, and showed that the **bounded approximation property** (BAP) of Banach spaces is preserved by Lipschitz isomorphisms [4]. The BAP is a linear property that enables us to approximate generic vectors by simpler 'finite-dimensional' ones, via a process that is somewhat algorithmic in nature. Stronger versions of the BAP (e.g. the **metric approximation property** (MAP), **finite-dimensional decompositions** (FDDs) and **Schauder bases**) yield improved approximation schemes.

Free spaces have since appeared in similar results, and are now objects of study in their own right. Despite their relatively simple definition, they are very complicated: $\mathcal{F}(\mathbb{R}) \equiv L_1$, and no explicit representation of $\mathcal{F}(\mathbb{R}^2)$ is known (though it cannot be linearly embedded as a subspace of L_1 [9]). Regarding applications, the canonical norm on $\mathcal{F}(M)$ is called the **Earthmover distance** in optimal transport theory and computer science, and $\mathcal{F}(\mathbb{R}^2)$ is used in image analysis [5]. In these fields, it is key to have effective algorithms to compute this distance.

2. Open problems

Despite having a simple definition, the linear structure of $\mathcal{F}(M)$ is complicated. Often, simple questions such as the presence or otherwise of approximation properties, (Schauder) bases or FDDs are open. The following natural question appears in [2].

Problem 1 *Given a subset M of $\mathbb{R}^N$, with metric induced by some norm on $\mathbb{R}^N$, does $\mathcal{F}(M)$ have the MAP?*

A partial positive solution to this problem is given in [11], however, it does not apply to any subset of $\mathbb{R}^N$ having empty interior, even ones that appear to be relatively simple such as

$$([0,1]\times\{0\})\cup\left\{(t,t^2)\ :\ t\in[0,1]\right\}\ \subseteq\ \mathbb{R}^2,$$

(the difficulty here lies at the cusp at $(0,0)$). The following is another problem in [2].

Problem 2 *Let M be a **uniformly discrete** metric space, that is, there exists $\theta > 0$ such that $d(x,y) > \theta$ for all distinct $x,y \in M$. Does $\mathcal{F}(M)$ have the BAP?*

This question was asked by Nigel Kalton. If the answer to Problem 2 is negative, it follows that there is an equivalent norm on ℓ_1 which fails to have the MAP (which would solve a 50-year-old problem). If the answer is positive, it follows that every separable Banach space is approximable, in the sense that the identity map is the pointwise limit of a sequence of equi-uniformly continuous functions having relatively compact range. Thus, either way, a solution to this problem would be significant. In this respect, [3] could yield important insights.

Turning now to bases and FDDs, the space $\mathcal{F}(\ell_1)$ has an FDD and moreover a basis [6, 8]. According to [7], if $M \subseteq \mathbb{R}^N$ has a non-empty interior then $\mathcal{F}(M)$ is isomorphic to $\mathcal{F}(\mathbb{R}^N)$, and thus has a basis (though the basis constant is generally not equal to 1, hence this result does not have any bearing on Problem 1). However, little more is known beyond that. The following natural questions arise.

Problem 3

1. *Let $M = \ell_2$ or $M = c_0$ with the usual norm. Does $\mathcal{F}(M)$ have a basis or FDD?*
2. *More ambitiously, given a separable Banach space M, does $\mathcal{F}(M)$ have a basis (or FDD) whenever M has a basis?*

Bases and FDDs are built using commuting families of finite-rank projections. Typically, these families are constructed by first restricting Lipschitz functions to their values on a finite lattice of points (thus ensuring that the resulting operators have finite rank), and then extending the restricted function back to the whole space. One main difficulty lies in the fact that extending Lipschitz functions in a **linear** manner is quite an intricate business, and it is only the properties of the usual norm on ℓ_1 that permit the 'obvious' extension method to work, in the sense that the norms of the extension operators remain bounded as the number of lattice points increases to infinity. We need new methods to construct such families to obtain bases or FDDs on ℓ_2 or c_0, as there is no clear way to adapt those developed in the $\mathcal{F}(\ell_1)$ case. Some recent work of Novotný has yielded Schauder bases on $\mathcal{F}(M)$, where $M \subseteq c_0$ is a certain infinite lattice [10].

References

[1] Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Vol. 1, Colloquium publications (American Mathematical Society) **48**, 2000.

[2] G. Godefroy, *A survey on Lipschitz-free Banach spaces*, Comment. Math. **55** (2015) 89–118.

[3] G. Godefroy, *Extensions of Lipschitz functions and Grothendieck's bounded approximation property*, North-Western European Journal of Mathematics **Vol. 1** (2015), 1 – 6.

[4] G. Godefroy and N. Kalton, *Lipschitz-free Banach spaces*, Studia Math. **159** (1) (2003) 121–141.

[5] L. J. Guibas, Y. Rubner, and C. Tomassi, *The earth mover's distance as a metric for image retrieval*, Int. J. Comput. Vision, **40** (2000) 99–121.

[6] P. Hájek and E. Pernecká, *On Schauder bases in Lipschitz-free spaces*, J. Math. Anal. Appl. **416** (2) (2014) 629–646.

[7] P. L. Kaufmann, *Products of Lipschitz-free spaces and applications*, preprint (`http://arxiv.org/abs/1403.6605`).

[8] G. Lancien and E. Pernecká, *Approximation properties and Schauder decompositions in Lipschitz-free spaces*, J. Funct. Anal. **264** (2013), 2323–2334.

[9] A. Naor and G. Schechtman, *Planar earthmover is not in* L_1, SIAM J. Comput. **37** (2007), 804–826.

[10] M. Novotný, Schauder Basis on Free-Lipschitz Spaces of Grids, poster, 2016.

[11] E. Pernecká and R. J. Smith, *The metric approximation property and Lipschitz-free spaces over subsets of* $\mathbb{R}^N$, J. Approx. Theory 199 (2015), 29–44

[12] N. Weaver, *Lipschitz Algebras*, World Sci., 1999.