3D rotated and standard staggered finite-difference solutions to Biot’s poroelastic wave equations: Stability condition and dispersion analysis

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ABSTRACT

A fourth-order in space and second-order in time 3D staggered (SG) and rotated-staggered-grid (RSG) method for the solution of Biot’s equation are presented. The numerical dispersion and stability conditions are derived using a von Neumann analysis. The exact stability condition is calculated from the roots of a 12th-order polynomial and therefore no nontrivial expression exists. To overcome this, a 1D stability condition is usually generalized to three dimensions. It is shown that in certain cases, the 1D approximate stability condition is violated by a 3D SG method. The RSG method obeys the approximate 1D stability condition for the material properties and spatiotemporal scales in the examples shown. Both methods have been verified against an analytical solution for an infinite homogeneous porous medium with a misfit error of less than 0.5%. A free surface has been implemented to test the accuracy of this boundary condition. It also serves as a test of the methods to include high material contrasts. The methods have been compared with a quasi-analytical solution. For the specific material properties, spatial grid scaling, and propagation distance used in the test, a maximum error of 3.5% for the SG and 4.1% for the RSG was found. These errors depend on the propagation distance, temporal and spatial scales, and accuracy of the quasi-analytical solution. No discernable difference was found between the two methods except for time steps comparable with the stability-criteria threshold time step, the SG was found to be unstable. However, the RSG remained stable for a homogeneous half-space. Time steps, comparable to the stability criteria, reduce the computational time at the cost of a reduction in accuracy. The methods allow wave propagation to be modeled in a porous medium with a free surface.

INTRODUCTION

Poroelasticity accounts for the interaction of elastic deformation of a material and viscous fluid flow in the same material. It has wide applications in engineering, physics, and the earth sciences. More specifically, in geophysics, poroelastic theory is needed when considering earthquake genesis, attenuation properties derived from seismic waves, reservoir modeling for oil recovery, and CO2 sequestration. Biot’s theory describes wave propagation in a saturated porous medium and accounts for the dissipation of energy due to the viscous pore fluid (Biot, 1962). A review of poroelastic theory can be found in Carcione (2007).

Biot’s equations in terms of the solid displacement $\mathbf{u}$ and fluid displacement relative to the solid $\mathbf{w}$ for an isotropic poroelastic medium are given by

$$\rho \ddot{\mathbf{u}} + \rho_f \ddot{\mathbf{w}} = (\lambda_c + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} + \alpha M \nabla \nabla \cdot \mathbf{w}$$  \hspace{1cm} (1)

and

$$\rho_f \ddot{\mathbf{w}} + m \ddot{\mathbf{w}} = \alpha M \nabla \nabla \cdot \mathbf{u} + M \nabla \nabla \cdot \mathbf{w} - b \mathbf{w}.$$  \hspace{1cm} (2)

The physical constants in equations 1 and 2 describing the poroelastic medium are listed in Table 1 along with some of their interdependencies. Few exact solutions to equations 1 and 2 exist and are only for simplified media and source terms (for example, infinite homogeneous porous media). Therefore, to apply Biot’s equations to a heterogeneous medium, numerical techniques must be employed. Several different numerical techniques can be used to solve equations 1 and 2, but in the present work, we focus solely on a finite-difference solution to Biot’s equations.

Finite-difference methods are a common numerical technique for solving differential equations (Press et al. 2007). Variables and constants in the equations are discretized onto a regular (or irregular) grid, and the spatial and temporal derivatives are replaced by finite-difference operators acting on variables at specific grid locations.

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Numerous finite-difference techniques have been applied successfully to the wave equation in different rheologies. Staggered grids (SG), where different material properties and variables are defined at different locations on the grid, are the most common stable finite-difference scheme for solving the wave equation. This approach allows the method to simulate a heterogeneous elastic solid with a large variation in Poisson’s ratio (Virieux, 1986). Saenger et al. (2000) introduces the rotated staggered grid (RSG) finite-difference method for modeling elastic waves. This approach allows the method to include cracks, pores, and free surfaces without explicitly accounting for them in the numerical method. The main advantage of the RSG is higher accuracy for high medium contrasts and highly anisotropic media. Finite-difference methods have been used to successfully model the viscoelastic wave equation (Emmerich and Korn, 1987, Robertsson et al., 1994, and Bohlen, 2002). The RSG method also has been applied successfully to viscoelastic and anisotropic wave propagation (Saenger and Bohlen, 2004). Also, Saenger et al. (2007) use an elastic RSG method to simulate the Biot slow wave in a microrock scale model where pores were filled with water or gas hydrates, modeled as an anelastic medium. Dai et al. (1995), Masson et al. (2006), and Sheen et al. (2006) solve poroelastic wave equations using a staggered-grid finite-difference approach. Wenzlau and Mueller (2009) implement 2D SG and RSG methods to solve poroelastic wave equations. In this paper, we describe 3D SG and RSG methods for a poroelastic medium and derive the stability condition and dispersion relations. We also compare the numerical method with an analytical solution and quasi-analytical solution for an infinite porous medium and a poroelastic half-space.

### ROTATED STAGGERED FINITE-DIFFERENCE METHOD

The set of equations solved in this study using finite-difference methods are derived from Biot’s equations for poroelastic medium, the stress-strain relationship for a porous medium, and the pressure in a porous medium (Biot, 1962). They are expressed in the velocity-stress formulation as

\[
(m \rho - P^2) i = m \tau_{ij} + p_1 b q_i + p_2 p_{ij},
\]

and

\[
\tau_{ij} = (\mu q_{ij}^2 + \nu_{ij}) + \delta_{ij}(\lambda q_{kk} + \alpha M q_{kk}).
\]

Here, \( \tau \) is the stress, \( p \) is pressure, solid velocity \( \mathbf{v} \) is the time derivative of \( \mathbf{u} \), and fluid velocity relative to the solid \( \mathbf{q} \) is the time derivative of \( \mathbf{w} \). A single source term or multiple source terms can be included in one or all of equations 3–6 as a solid stress, fluid pressure, or solid and/or fluid forces. The exact 3D stability condition and dispersion relation have not been derived yet for either the SG or RSG method. Therefore, the stability and the effect of numerical dispersion on the solution have yet to be fully quantified. At low frequencies (typically the seismic band), a wave propagating across a porous medium will create laminar flow in the pore space, implying that viscous boundary layers need not be considered. At higher frequencies, this effect cannot be ignored but is not considered in the implementations discussed above. Biot (1956a, 1965b) provides the theory of elastic wave propagation in porous media for low- and high-frequency ranges. We also will not consider the viscous boundary layer because it is not large for the seismic frequency band.

From equations 3–6, we have four variables \( \mathbf{v}, \mathbf{q}, \tau, \) and \( p \) and eight material parameters \( \rho, \rho_1, m, b, \mu, \lambda, M, \) and \( \alpha \) which are distributed on the grid depending on the finite-difference implementation. Distribution of the parameters needed is shown in Figure 1 for both implementations. In the staggered grid, velocity components are located at different points, with each being half a grid point from the corner in the appropriate component direction. Normal stresses are located on the corner and shear stresses are located in the center of the appropriate plane. For the rotated staggered grid, stress and pressure are located on the node corners and solid and fluid velocities are located on the node midpoints. In both methods, all material parameters are located on the node corners. Derivative directions for the RSG are also shown in Figure 1 and are described in Saenger et al. (2000). We use a fourth-order in space and second-order in time implementation with \( \Delta x, \Delta y, \) and \( \Delta z \) being the spatial grid steps.

<table>
<thead>
<tr>
<th>Name</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho, \rho_1 )</td>
<td>kg/m(^3)</td>
</tr>
<tr>
<td>( \rho )</td>
<td>kg/m(^3)</td>
</tr>
<tr>
<td>( \phi )</td>
<td>kg/m(^3)</td>
</tr>
<tr>
<td>( m )</td>
<td>kg/m(^3) ( P \phi )</td>
</tr>
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<td>( \Phi )</td>
<td>kg/m(^3) ( T \phi )</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Pa s ( \eta / \kappa )</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>m(^3) ( \eta / \kappa )</td>
</tr>
<tr>
<td>( T )</td>
<td>m ( \eta / \kappa )</td>
</tr>
<tr>
<td>( b )</td>
<td>Pa s/m(^2) ( \eta / \kappa )</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Pa ( \lambda + \alpha^2 M )</td>
</tr>
<tr>
<td>( \lambda_c )</td>
<td>Pa ( \lambda + \alpha^2 M )</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Pa ( \lambda + \alpha^2 M )</td>
</tr>
<tr>
<td>( M )</td>
<td>Pa ( K_s (1 - \Phi - K_s / K_i + \Phi K_s / K_i)^{-1} )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Pa ( 1 - K_s / K_i )</td>
</tr>
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<td>( K_i )</td>
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</tr>
<tr>
<td>( K_i )</td>
<td>Pa ( \lambda + 2/3 \mu )</td>
</tr>
</tbody>
</table>

Table 1. Some material parameters needed to describe a porous medium. The units and some interdependencies are also listed, and \( \rho, \rho_1, m, \lambda, \mu, M, \) and \( \alpha \) are the material-parameter inputs in the RSG finite-difference method.
and $\Delta t$ is the temporal time step. Equations 3–6 then can be written in discretized form for the RSG method as

$$v^t + \frac{\Delta v^t}{2}(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2) = v^{t-\frac{\Delta v^t}{2}}(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2) + \left\{ \frac{\Delta t}{\bar{m} \bar{\rho} - \bar{\rho}^2} \right\} \times (\bar{m} D_i \tau_{ij}(x,y,z) + \bar{p}_i \bar{b} \bar{q}_j(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2) + \bar{p}_j D_i p(x,y,z)), $$

$$q^t + \frac{\Delta q^t}{2}(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2) = q^{t-\frac{\Delta q^t}{2}}(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2) - \left\{ \frac{\Delta t}{\bar{m} \bar{\rho} - \bar{\rho}^2} \right\} \times (\bar{p}_i D_j \tau_{ij}(x,y,z) + \bar{p}_j \bar{b} \bar{q}_i(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2) + \Delta z/2) + \bar{p}_j D_i p(x,y,z)), $$

$$r^t + \frac{\Delta r^t}{2}(x,y,z) = r^{t-\frac{\Delta r^t}{2}}(x,y,z) + \Delta t \mu [D_i \nu_k(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2) + D_i \nu_k(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2)] + \Delta t \delta_{ij} \lambda D_i \nu_k(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2)), $$

$$p^t + \frac{\Delta p^t}{2}(x,y,z) = p^{t-\frac{\Delta p^t}{2}}(x,y,z) - \Delta t (\alpha M D_k \nu_k(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2) + \alpha M D_k \nu_k(x + \Delta x/2,y + \Delta y/2,z + \Delta z/2))), $$

In equations 7–10, the tilde over variables indicates a time average:

$$\bar{q}_i = \frac{1}{2} (q^t_i + \Delta q^t/2 + q^{t-\Delta q^t/2}).$$

The corresponding equations for the SG method can be derived easily from equations 7–10 by placing the variables in their corresponding SG locations. Source terms can be added to equations 7–10 above by the system of equations 2

$$\omega_B = \frac{b \phi}{T \rho_f}.$$  

When the friction is decreased or the source frequency is sufficiently increased, the slow P-wave becomes propagative. However, as discussed above, the viscous layers in a real material would become important, but this effect can be omitted for the seismic frequency band. For the sake of completeness, several examples are shown in both regimes. However, it must be remembered that the above equations should include the viscous boundary layers in the high-frequency regime.

**DISPERSION ANALYSIS AND STABILITY CONDITION**

The accuracy and stability of any finite-difference numerical scheme depend heavily on 1) the system of equations, 2) finite-difference operators, 3) order of the finite-difference operators, 4) spatial and temporal grid steps, and 5) material contrasts. To quantify dispersion and stability of the finite-difference solutions of Biot’s equations, we perform a von Neumann analysis. Substituting the displacement-stress formulation of equations 5 and 6 into the displacement-stress formulation of equations 3 and 4, the equations then become

$$(mp - \rho f^2) D_i \nu_i - m D_j \mu (D_i \mu_j + D_i \mu_j) + \delta_{ij} \lambda D_k \nu_k + \alpha M D_k \nu_k - \rho_j D_i (\alpha M D_k \nu_k + MD_k \nu_k) = 0.$$  

and

$$(mp - \rho f^2) D_i \nu_i + \rho D_j (\mu (u_i, u_j) + u_i, u_j) + \delta_{ij} \lambda D_k \nu_k + \alpha M D_k \nu_k + \rho b D_i \nu_i - \rho (\alpha M D_k \nu_k + MD_k \nu_k) = 0.$$  

Equations 13 and 14 can be reformulated in matrix form as

![Figure 1. Grid layout of the SG and RSG finite-difference methods for the solution of Biot’s equations for wave propagation in a porous medium. The length of the cube’s side is $\Delta t$.](image-url)
\[ Q \cdot U = 0, \quad (15) \]

where
\[
Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix},
\]

(16)

with
\[
Q_1 = \begin{bmatrix} a_{11}D_{xx} + b_{11}D_{xy} + c_{11}D_{xz} & a_{12}D_{xx} + b_{12}D_{xy} + c_{12}D_{xz} \\ a_{21}D_{xx} + b_{21}D_{xy} + c_{21}D_{xz} & a_{22}D_{xx} + b_{22}D_{xy} + c_{22}D_{xz} \end{bmatrix}.
\]

(17)

\[
Q_2 = \begin{bmatrix} a_{31}D_{xx} + b_{31}D_{xy} + c_{31}D_{xz} & a_{41}D_{xx} + b_{41}D_{xy} + c_{41}D_{xz} \\ a_{32}D_{xx} + b_{32}D_{xy} + c_{32}D_{xz} & a_{42}D_{xx} + b_{42}D_{xy} + c_{42}D_{xz} \end{bmatrix},
\]

(18)

\[
Q_3 = \begin{bmatrix} a_{31}D_{xx} + b_{31}D_{xy} + c_{31}D_{xz} & a_{41}D_{xx} + b_{41}D_{xy} + c_{41}D_{xz} \\ a_{32}D_{xx} + b_{32}D_{xy} + c_{32}D_{xz} & a_{42}D_{xx} + b_{42}D_{xy} + c_{42}D_{xz} \end{bmatrix},
\]

(19)

\[
Q_4 = \begin{bmatrix} a_{31}D_{xx} + b_{31}D_{xy} + c_{31}D_{xz} & a_{41}D_{xx} + b_{41}D_{xy} + c_{41}D_{xz} \\ a_{32}D_{xx} + b_{32}D_{xy} + c_{32}D_{xz} & a_{42}D_{xx} + b_{42}D_{xy} + c_{42}D_{xz} \end{bmatrix},
\]

(20)

and
\[
U = [u_x \ u_y \ u_z \ w_x \ w_y \ w_z]^T.
\]

(21)

Material variables \(a_{ij}, b_{ij}, \) and \(c_{ij}\) in the matrix \(Q\) are given in the appendix. The von Neumann analysis assumes a plane-wave solution of the form \(u(x,y,z,t) = Ae^{-i(k_x x + k_y y + k_z z)}\) with wavenumbers \(k_x, k_y,\) and \(k_z,\) and angular frequency \(\omega.\) By allowing the derivatives in equations 13 and 14 to act on the plane wave and using the operators defined in the appendix, the SG derivative terms in the matrix \(Q\) become
\[
D_{xx} = -\frac{4}{\Delta x^2} \left[ c_1^2 \sin^2 \left( \frac{k_x \Delta x}{2} \right) \cos^2 \left( \frac{k_y \Delta x}{2} \right) \cos^2 \left( \frac{k_z \Delta x}{2} \right) \right]
\]

\[
\times \left[ -\frac{4}{\Delta x^2} \left( 3k_x \Delta x \right)^2 \cos^2 \left( \frac{k_y \Delta x}{2} \right) \cos^2 \left( \frac{k_z \Delta x}{2} \right) \right]
\]

\[
\times \left( \cos^2 \left( k_y \Delta x \right) - \sin^2 \left( \frac{k_y \Delta x}{2} \right) \right)
\]

\[
\times \left( \cos^2 \left( k_z \Delta x \right) - \sin^2 \left( \frac{k_z \Delta x}{2} \right) \right) \quad (24)
\]

and
\[
D_{xz} = \frac{-1}{2\Delta x^2} \left[ c_1^2 \sin \left( k_x \Delta x \right) \sin \left( k_z \Delta x \right) \left( 1 + \cos \left( k_y \Delta x \right) \right) \right]
\]

\[
\times \left[ -\frac{1}{2\Delta x^2} \left( c_1^2 \sin \left( 3k_x \Delta x \right) \sin \left( 3k_z \Delta x \right) \right) \left( 1 + \cos \left( 3k_y \Delta x \right) \right) \right]
\]

\[
- \sin \left( k_x \Delta x \right) \sin \left( k_z \Delta x \right) \left( \cos \left( 2k_y \Delta x \right) + \cos \left( k_y \Delta x \right) \right) \right].
\]

(25)

The other derivatives in matrix \(Q\) can be expressed in a similar manner by exchanging the appropriate spatial labels \(x, y,\) and \(z.\) The matrix \(Q\) is then dependent on the wavenumbers and \(D_i\) and the determinant \(|Q|\) can be written as:
\[
|Q| = \sum_{n=0}^{12} p_n \left( i \sin \left( \omega \Delta t \right) / \Delta t \right)^n
\]

(26)

Here, \(p_n\) are dependent on \(a_{ij}, b_{ij}, c_{ij},\) and \(D_i\) in matrix \(Q\) and contain up to several hundred terms. The sine function appears by allowing \(D_i\) to act on the plane wave solution. For the schemes to be unconditionally stable, \(|Q|\) is less than 1 for all wavenumbers. Equation 26 can be used to determine a stable time step given the material parameters and spatial grid step. A solution to equation 26 for \(|Q| \leq 1\) can be found by using a Newton method or a mathematical package provided that the constants \(p_n\) have been predetermined. This is nontrivial because these constants contain up to several hundred terms and are not easily determined. Therefore, to find a stable time step without knowing \(p_n,\) the 1D stability criterion is given below and has been adjusted to a 3D scheme by multiplying by \(1/3.\) (Masson et al., 2006). The approximate unconditionally stable time step \(\Delta t_s\) is given by
\[
\Delta t_s \leq \frac{1}{3} \left( \frac{\Delta x}{c_1 - c_2} \right) \left( \frac{s_2 - (s_2^2 - 4s_1s_3)^{1/2}}{2s_3} \right),
\]

(27)

where
\[
s_1 = mp - \rho_t^2,
\]

(28)
\[ s_2 = m(\lambda_c + 2\mu) + \rho M - 2\alpha M \rho_f, \quad (29) \]

and
\[ s_3 = M \lambda + 2\mu M - \alpha^2 M^2. \quad (30) \]

This 1D stability condition should be used only if the exact stability condition determined from equation 26 cannot be computed directly and (as seen in Figure 2) it can be violated for the SG method. As discussed in Masson et al. (2006), a material stability condition also exists and can be understood from equations 7 and 8. The division by \( s_1 \) (defined in equation 28) leads to an infinity if \( s_1 = 0 \) and, therefore, gives the material stability condition \( s_1 \neq 0 \).

Determinant \( |Q| \) has been calculated for material parameters listed in Table 2 for model 1. The time step was chosen as \( \Delta t = 0.5\Delta t_s, \Delta t = 0.75\Delta t_s, \Delta t = \Delta t_s, \) and \( \Delta t = 1.25\Delta t_s \) with \( k_x = k_y = k_z \) for wavenumbers in the range \( [0, \pi/\Delta x] \). Figure 2 plots the value of the determinant for each wavenumber, and for the scheme to be stable, all values must lie within the complex unit circle. As seen from Figure 2c, the 1D stability criterion from Masson et al. (2006) ensures the scheme is unconditionally stable for all wavenumbers for the RSG method. In fact, the scheme is stable for \( \Delta t = 1.25\Delta t_s \) for these specific material properties. However, the SG method is only stable for \( \Delta t = 0.5\Delta t_s \). Wavenumbers that violate the stability condition correspond to small wavelengths, which would not be allowed because they would suffer large numerical dispersion. The violation of the stability condition can be viewed as an increase in the phase velocity for small wavelengths due to dispersion effects. Dispersion relations for the different phases are derived from equation 26 by solving it for \( \sin^{-1}(k) \). Figure 3 shows the dispersion of the fast P-wave, slow P-wave, and S-waves for model 1 in Table 2 for three time steps and three propagation directions. The theoretical low-frequency-limit phase velocities calculated from equations 7.288 and 7.311 in Carcione (2007) also are plotted in the figure. The increase in phase velocity is seen clearly in the SG case. Figure 4 shows the dispersion relations and stability circles for models 2, 3, and 4 in Table 2. The effect of the friction can be clearly seen in the difference between Figure 4b and c, where the slow P-wave is not propagative in the high-friction model.

### COMPARISON WITH ANALYTICAL SOLUTIONS

**Infinite homogeneous porous medium**

The SG and RSG methods described above were tested against an analytical solution for wave propagation in a porous medium. The analytical solution is given in Dai et al. (1995) for a homogeneous porous medium with an isotropic point source. In this case, the source was implemented as an isotropic stress in equation 5. Material parameters (model 1) used for the simulation are listed in Table 2. The analytical solution used has \( b = 0 \), hence there is no damping included. Although \( b = 0 \) represents an inviscid fluid in a porous media, it allows us to verify the method against the exact analytical solution. The spatial grid step was 2.5 m and the time step was set at \( 5 \times 10^{-4} \) s. The synthetic receivers were spaced 25-m apart for a Ricker wavelet source with a central frequency of 30 Hz. Numerical and analytical solutions are compared in Figure 5 for the solid and fluid velocities for both methods. A visual inspection clearly shows that the solutions match extremely well with no visible difference between the SG and RSG solutions. However, to quantify the comparison between the analytical and numerical signals, we computed the misfit \( \xi \) given by

\[ \xi = \frac{\sum (S_{NUM}(t) - S_{ANA}(t))^2}{\sum S_{ANA}(t)^2}. \quad (31) \]

Here, \( S_{ANA}(t) \) is the analytical signal and \( S_{NUM}(t) \) is the numerical signal. The misfits for both the solid and fluid velocities in Figure 6 show that both the numerical methods give a very good fit with the analytical solution. The solid radial velocity is dominated by the fast P-wave and the fluid radial velocity is dominated by the slow P-wave. Both components show that the fast P-wave and slow P-wave are well resolved by the numerical method. As expected, the misfit increases with distance from the source as the effects of numerical dispersion increase with numerical time and distance. Periodic boundary conditions were imposed in this example and all other examples except where a free surface is used. Implementing absorbing boundaries for a poroelastic finite-difference method is discussed in Sheen et al. (2006).
Table 2. Material constants for different media used in this study.

<table>
<thead>
<tr>
<th></th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
<th>Model 5</th>
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<td>(\Delta x) (m)</td>
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</tbody>
</table>

Figure 3. Dispersion curves for the fast P-wave, slow P-wave, and S-wave for model 1 in Table 2. Dispersion curves for (a) rotated-staggered-grid method for three different time steps: solid line \(\Delta t = 0.5\Delta t\), dashed line \(\Delta t = 0.75\Delta t\), and dot-dash line \(\Delta t = 0.95\Delta t\); (b) staggered-grid method for the same three different time steps; (c) rotated-staggered-grid for three different propagation directions: solid line \(\mathbf{k} = [0, 0, 1]\), dashed line \(\mathbf{k} = [0, 1, 1]\), and dot-dash line \(\mathbf{k} = [1, 1, 1]\), where \(\mathbf{k} = [k_x, k_y, k_z]\). (d) Staggered grid for the same three propagation directions. Theoretical low-frequency-limit phase velocities (calculated from Carcione, 2007) are also plotted on both subplots as the horizontal lines, for the fast P-wave, S-wave, and slow P-wave from top to bottom, respectively.

Figure 4. Dispersion curves calculated for three different porous media and the associated stability circles. Material parameters are given for the three media in Table 2: (a) model 2, (b) model 3, and (c) model 4. Solid lines and crosses are for the RSG method and dashed lines and circles are for the SG solution. Low-frequency-limit phase velocities are plotted as horizontal lines for the fast P-wave, S-wave, and slow P-wave from top to bottom, respectively. In (b), no propagative slow P-wave exists because it is diffusive at this scale.
roelastic half-space (Philippacopoulos, 1997, 1998). The freq\-uency-
domain analytical solution requires an inverse Hankel transforma-
tion, which is solved numerically (Guizar-Sicairos and Gutierrez-
Vega, 2004). Then the solution is transformed into the time domain
by an inverse Fourier transform. Recorders were spaced 15-m apart
inside the medium and on the free surface for a 40-m deep source. As
opposed to the previous analytical solution, we have a nonzero $b$
value so attenuation is included. The source function is a Ricker wavelet
vertical force with a central frequency of 30 Hz and the material pa-
rameters correspond to model 5 in Table 2. It should be noted that for
these material properties the slow P-wave is diffusive. The vertical
force was included in equations 3 and 4 with a multiplication factor
of $m$ and $f$ for the solid and fluid respectively.

Quasi-analytical and numerical solutions are compared in Figure
6 along with the misfit for three different time steps for the SG and
RSG methods. As the time step is decreased from 0.25 ms to 0.05 ms
to 0.025 ms, the solution converges to the quasi-analytical solution.
No solution is plotted for the SG method with a time step of 0.25 ms
because the scheme becomes unstable. For the remaining simula-
tions, little difference is observed in the misfit between the two nu-
umerical methods. It should be noted that numerical errors also exist
in the quasi-analytical solution because the numerical implementa-
tion of the inverse Hankel transform will induce errors in the solu-
tion. Comparison of the analytical and numerical fluid velocity
shows similar results. The final test is shown in Figure 7, where we
have replicated the same setup but have significantly decreased the
friction to allow the slow P-wave to propagate (model 4 in Table 2).
Figure 7 shows the comparison of the seismograms in the medium
and on the free surface along with the misfits for three time steps. As
in the previous case, no significant difference is observed between
the SG and RSG methods except for the 0.25-ms time step where the
SG method became unstable.

![Figure 5. Comparison of an analytical solution (solid line) and the
SG (dash-dot lines) and RSG methods (dashed line) for wave propa-
gation in an infinite homogeneous porous medium. There is no visu-
al difference between the solutions on the scale presented. Solid dots
linked by the solid line show the misfit for the RSG method. Solid tri-
gle linked by the dashed line show the misfit for the SG method.
Material parameters for the homogenous medium are listed in Table
2 for model 1. All waveforms, which were measured at 25-m inter-
vals, have been normalized with respect to the analytical solution.
(a) The fast P-wave is seen clearly for the solid radial velocity, and
(b) the fluid radial velocity is dominated by the slow P-wave.]

![Figure 6. (a) Comparison of a quasi-analytical solution (solid line)
and RSG method (dashed line) for wave propagation in a homoge-
eous porous half-space, model 5. Lower seismograms are located
inside the medium and upper seismograms are located on the free
surface. To clearly see all plotted waveforms, the analytical and nu-
umerical solutions have been normalized using the maximum ampli-
tude of the analytical solution for each individual trace. (b) Misfits
between the analytical and RSG numerical solution are shown for
each individual trace. The time steps were 0.25, 0.05, and 0.025 ms
for the squares, triangles, and circles respectively. (c) The SG solu-
tion compared with the analytical solution. Time steps were 0.05 ms
and 0.025 ms for the squares and circles, respectively. No solution
exists for the SG method with a time step of 0.25 ms because it was
unstable.]
Largest material contrasts in any geological formation, and thus the space. The free-surface boundary condition represents one of the provides a good solution for wave propagation in a poroelastic half-
results with a quasi-analytical solution. We show that the method pro-
a homogeneous porous medium. In both methods, we have included a free-surface boundary condition and compared our numerical re-
expression for the stability condition and dispersion curves. The use of a 1D approximate stability condition is not advised as the SG violates the stability condition and dispersion curves. However, in seismology, the lack of detailed velocity models usually would be responsible for the largest error associated with computational wave propagation.

CONCLUSIONS

We have implemented fourth-order in space and second-order in time staggered-grid and rotated-staggered-grid methods for the solution of Biot’s equation for wave propagation in a porous medium. Biot equations are stiff and care must be taken when choosing the scale of the problem because the Biot slow wave behaves diffusively at low frequency and becomes propagative at high frequencies. Also, without correcting for the viscous boundary layer at high frequencies, the method should be restricted to the low-frequency regime, i.e., $\omega \ll \omega_0$. The exact 3D numerical dispersion and stability conditions of the methods were derived using a von Neumann analysis. Determination of the required time step depends on the evaluation of a determinant of a $6 \times 6$ matrix. This matrix depends in a nontrivial way on the spatial grid step, the time step, and material parameters, as well as the wavenumber. This prohibits the derivation of a simple expression for the stability condition and dispersion curves. The use of a 1D approximate stability condition is not advised as the SG violates it for specific material properties. The numerical methods have been verified against an analytical solution for an isotropic source in a homogeneous porous medium. In both methods, we have included a free-surface boundary condition and compared our numerical results with a quasi-analytical solution. We show that the method provides a good solution for wave propagation in a poroelastic half-space. The free-surface boundary condition represents one of the largest material contrasts in any geological formation, and thus the schemes are applicable to high medium contrasts once sufficiently small spatial and temporal scales are selected. The RSG method stays stable for a larger time step than the SG method, which reduces computational time but comes at the cost of a reduction in accuracy. For sufficiently small time steps, no discernable difference between the methods is observed for the examples illustrated here. It should be noted that accuracy of the methods depends on the propagation distance, temporal grid step, spatial grid step, and material properties. Therefore, misfit errors calculated for the models used here could increase or decrease equally depending on the computational resources and model properties. Selection of an appropriate scale to minimize errors can be determined from the exact 3D stability conditions and dispersion curves. However, in seismology, the lack of detailed velocity models usually would be responsible for the largest error associated with computational wave propagation.

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APPENDIX A

The staggered-grid first and second time and spatial derivatives are given by

$$\frac{\partial u}{\partial t} = D_u \frac{u^{t+At/2} - u^{t-At/2}}{At} ,$$  \hspace{1cm} (A-1)

$$\frac{\partial^2 u}{\partial t^2} = D_u \frac{u^{t+At} - 2u^t + u^{t-At}}{At^2} ,$$ \hspace{1cm} (A-2)

$$\frac{\partial u}{\partial x} = D_u \frac{u^{x+\Delta x/2,y,z} + u^{x-\Delta x/2,y,z}}{\Delta x} + \frac{c_x}{\Delta x} \left[ u(x+\Delta x,y,z) + u(x,y,z) \right]$$

$$+ \frac{c_z}{\Delta x} \left[ u(x, y + \Delta y, z) + u(x, y - \Delta y, z) \right] .$$  \hspace{1cm} (A-3)

The other derivative directions are calculated in a similar manner to $\partial u/\partial x$. The RSG spatial derivatives are given by

$$\frac{\partial u}{\partial x} = D_u \frac{u^{x+\Delta x/2,y,z} + u^{x+\Delta x/2,y+\Delta y/2} + u^{x+\Delta x/2,y-\Delta y/2}}{4\Delta x}$$

$$= \frac{1}{4\Delta x} \left[ c_1 (u(x+\Delta x,y,z) + u(x+\Delta x,y+\Delta y) + u(x+\Delta x,y-\Delta y) - u(x,y,z) - u(x,y+\Delta y,z) - u(x,y-\Delta y,z) + u(x+\Delta x,y+\Delta y) + u(x+\Delta x,y-\Delta y) - u(x,y,z) + u(x,y+\Delta y,z) + u(x,y-\Delta y,z) \right]$$

where $c_1$ and $c_2$ are defined as $$c_1 = \frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} \right)$$ and $$c_2 = \frac{1}{2} \left( \frac{\partial^2 u}{\partial y^2} \right).$$
\[
\frac{\partial u}{\partial y} = D_y u(x, y, z) \left[ (x + \Delta y) \right] \]

\[
\frac{\partial u}{\partial x} = D_x u(x, y, z) \left[ (x + \Delta x) \right] \]

Constants \(c_1\) and \(c_2\) are 9/8 and \(-1/24\), respectively (Dablain, 1986). Material variables \(a_{ij}, b_{ij}\), and \(c_{ij}\) in matrix \(Q\) in equation 16 are given by

\[
a_{11} = mp - \rho_{\gamma}, \quad \text{(A-7)}
\]

\[
b_{11} = -2\rho\mu + \rho_{\gamma}\alpha M - m\lambda, \quad \text{(A-8)}
\]

\[
c_{11} = -m\mu, \quad \text{(A-9)}
\]

\[
a_{21} = -m\lambda - \rho_{\gamma}aM, \quad \text{(A-10)}
\]

\[
a_{41} = -\rho_{\gamma}b, \quad \text{(A-11)}
\]

REFERENCES


